.

COMMUTATOR BOUNDS FOR EIGENVALUES, WITH APPLICATIONS TO SPECTRAL GEOMETRY

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Abstract

We prove a purely algebraic version of an eigenvalue inequality of Hile and Protter, and derive corollaries bounding differences of eigenvalues of Laplace–Beltrami operators on manifolds. We significantly improve earlier bounds of Yang and Yau, Li, and Harrell.

1 Introduction

In a recent paper [6], one of us produced an algebraic version of a well–known bound on differences between eigenvalues due to Payne, Pólya, and Weinberger [18]. They had shown that the difference of successive eigenvalues of the Dirichlet

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Laplacian on a domain in $\mathbf{R}^\nu$ is bounded by a universal constant times the sum of all the lower eigenvalues. It is easy to see, however, that if the analogous problem is considered on a manifold, then the geometry must enter into the relationship between the differences and the sum, and [6] showed specifically that the relationship depends in a certain way on the sectional curvature of the manifold. The algebraic bound in that article involves auxiliary operators, which can be specially chosen to reveal the geometric content of the eigenvalue differences. The philosophy of this article will be the same, except that we shall prove a different algebraic bound, which allows sharper estimates.

A reasonable way to frame the problem is as follows. Consider a Riemannian manifold $\mathcal{M}$, with subdomains $\mathbf{\Omega}$. The Laplace–Beltrami operator acts on smooth scalar functions on $\mathbf{\Omega}$ by $\Delta f = \nabla \cdot \nabla f$, where $\nabla \cdot$ and ∇ are the covariantly defined divergence and gradient. In local coordinates it has the form

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_i \, g^{ij} \sqrt{g} \, \partial_j f \tag{1.1}$$

and it is defined as a self–adjoint operator on appropriate Sobolev spaces incorporating the boundary conditions. Here g is the determinant of the metric tensor g_{ij}, and g^{ij} is the contravariant (inverse) metric tensor. For details on Laplace–Beltrami eigenvalue problems, we refer the reader to [5], [4], and [17].

The Dirichlet eigenvalues of the Laplace–Beltrami operator on a given domain will be denoted λ_ℓ, $\ell = 1, 2, \ldots, n$; these eigenvalues form a sequence of positive numbers accumulating at infinity. Let $\langle f_\ell \rangle_{\ell \le n}$ denote the average of an expression involving eigenvalues over all $\ell \le n$. Thus, for example $\langle \lambda_\ell \rangle_{\ell \le n} = \frac{1}{n} \sum_{\ell \le n} \lambda_\ell$. If, as will sometimes occur below, $\ell = 0, 1, \ldots, n$, we divide by $n+1$. We now define a constant which will turn out to reflect the geometry of $\mathcal{M}$ by

$$C_{PPW}(\mathcal{M}) := \sup_{n, \mathbf{\Omega} \subset \mathcal{M}} \frac{\lambda_{n+1} - \lambda_n}{\langle \lambda_\ell \rangle_{\ell \le n}} .$$

In the original article of Payne, Pólya, and Weinberger [18], the manifold $\mathcal{M}$ was Euclidean, and $C_{PPW} = \frac{4}{\nu}$, where $\nu :=$ dimension of $\mathcal{M}$ was actually fixed at 2. Many extensions of this have been made; the most up–to–date survey of the situation is [2]. For other related work and background see [7], [19], and [5]. Since $1 + C_{PPW}$ is an upper bound on the ratio λ_2/λ_1, and this ratio is arbitrarily large in the geometric setting, C_{PPW} is generally more complicated on manifolds. For example, the n-sphere with a small cap removed will have λ_1 close to 0, but λ_2 is bounded away from 0.

Several people (e.g. [14], [15], [21]) have studied bounds on gaps and ratios of eigenvalues to see how Riemannian geometry is revealed in the analysis of the

Laplace–Beltrami operator. Most progress has occured in the context of spaces of high symmetry; one of our goals here is to sharpen several of the more significant of these bounds. Following [6] we also produce some bounds for fairly general Riemannian manifolds.

One of the extensions of [18], due to Hile and Protter [10] at first sight may seem to be a mere technical improvement, since it has a more complicated relationship between the eigenvalue differences and the sums, which reduces to that of Payne, Pólya, and Weinberger in what is usually the most interesting case, that of the gap between λ_2 and λ_1. The proof seems to follow the steps of the earlier result for the most part, except for the order of the steps and the introduction of free parameters (as it turns out, unnecessarily). In our abstract version of the Hile–Protter inequality, the underlying algebra is actually somewhat distinct from the PPW bound, and in addition it allows more flexibility because there are two families of auxiliary operators, as opposed to the one in [6]. These differences are not important for the original problem with a Euclidean Laplacian, but are quite helpful in the situations we consider below.

Earlier, Hook produced an abstract algebraic bound of the Hile–Protter type and made several applications, recovering as special cases several prior bounds on various operators on domains in $\mathbf{R}^\nu$ (see [11] and [12]). Hook followed the original argument of Hile and Protter rather closely, and in particular, his result contains the same free parameter as in [10]. In order to recover the prior bounds, he makes special choices of the operators in the abstract theorem and optimizes over the parameter. Our Theorem 2.1 is similar in nature, but avoids the parameter altogether, and recovers a variety of results directly with special choices of the abstract operators. Some non-geometric applications of this technique are dealt with in a separate paper [8].

The Hile–Protter inequality is usually expressed as a lower bound, but for comparison with the PPW inequality, we define a comparable constant:

$$C_{HP}(\mathcal{M}) := \sup_{n,\boldsymbol{\Omega}\subset\mathcal{M}} \left\langle \frac{\lambda_\ell}{\lambda_{n+1}-\lambda_\ell} \right\rangle_{\ell\leq n}^{-1}.$$

Observe that $C_{HP} \geq C_{PPW}$, so for upper bounds on eigenvalue differences, it is not necessary to distinguish the two constants.

2 A theorem for abstract operators on Hilbert space

The operator under study is a self–adjoint operator H, and there are two families of symmetric "test operators," which we call G_j and Π_j. (The Π_j's are often analogues of the momentum operator of quantum mechanics, accounting for our notation. A rough correlation with Hook's notation is that our G's correspond to his B's and our Π's correspond to his T's times i.)

Theorem 2.1 *Let H be self-adjoint on a Hilbert space $\mathcal{H}$, and suppose that the lower portion of its spectrum consists of discrete eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < \lambda_{n+1} \leq \ldots$ Let $P_{\leq n}$ be the spectral projection for $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let $\{G_j\}$ and $\{\Pi_j\}$ be two families of symmetric operators such that all products of the form $\Pi_j G_j$, $G_j \Pi_j$, $G_j^2 H$, $H G_j^2$, and $G_j H G_j$ are well defined. Then*

$$\sum_{j=1}^{m} Tr\left(\left(\lambda_{n+1} I - H\right)^{-1} P_{\leq n} \Pi_j^2\right) \geq \frac{\left|\sum_{j=1}^{m} Tr\left(P_{\leq n}\left[\Pi_j, G_j\right]\right)\right|^2}{2\sum_{j=1}^{m} Tr\left(P_{\leq n}\left[G_j, [H, G_j]\right]\right)}, \tag{2.1}$$

assuming that these three traces are finite and nonzero.

Remarks: The operator $\lambda_{n+1} I - H$ is uniquely invertible as a positive operator from the range of $P_{\leq n}$ to itself.

The natural setting for this theorem is that of C^* algebras, in which the assumption on products of operators is unnecessary. Here, however, we are interested in unbounded operators, so domain questions must be considered carefully. While there is a certain amount of freedom in the choice of the auxiliary operators Π_j and G_j, it is important that the G_j's are chosen in such a way that HG_j is defined on the given domain, i.e. for $u \in \mathcal{D}(H)$ we must have $G_j u \in \mathcal{D}(H)$. Similarly for $G_j^2 u$.

In many applications the operators Π_j are chosen so that, in the quadratic–form sense,

$$\sum_{j=1}^{m} \Pi_j^2 \leq \beta H$$

for some constant β (which could be scaled to 1). Thus, in terms of the constants defined in the introduction,

$$C_{PPW} \leq C_{HP} \leq \frac{2n\beta \sum_{j=1}^{m} Tr\left(P_{\leq n}[G_j, [H, G_j]]\right)}{\left|\sum_{j=1}^{m} Tr\left(P_{\leq n}[\Pi_j, G_j]\right)\right|^2}. \tag{2.2}$$

Most frequently, the auxiliary operators will be chosen to satisfy

$$\Pi_j = i[H, G_j] = i(-\Delta G_j - 2\nabla G_j \cdot \nabla). \tag{2.3}$$

The bounds that result with this specialization closely resemble those of [6], but improve them a bit. This is because in this case,

$$\frac{2n\sum_{j=1}^{m} Tr\left(P_{\leq n}[G_j,[H,G_j]]\right)}{\left|\sum_{j=1}^{m} Tr\left(P_{\leq n}[\Pi_j,G_j]\right)\right|^2} = \frac{2n}{\sum_{j=1}^{m} Tr\left(P_{\leq n}[G_j,[H,G_j]]\right)}. \tag{2.4}$$

We recall two familiar properties of the trace, which will be used frequently, without comment, in the proof of the theorem:

1. the cyclic property of the trace: $Tr(AB) = Tr(BA)$,
2. $Tr(A^*B)$ is an inner product on A and B. In particular, the Cauchy–Schwarz inequality holds in the form: $|Tr(A^*B)|^2 \leq Tr(A^*A)\, Tr(B^*B)$.

Since we are taking traces of products of operators, each product contains a finite projection, so there are no convergence difficulties. We also use the fact that spectral projections commute with H and with one another.

The proof will be given as a series of simple lemmas.

Lemma 2.2 *Let P be a finite rank orthogonal projection, $Q := 1 - P$, and let G and Π be symmetric. Then,*

$$\Im m\left(Tr(P\Pi G)\right) = \Im m\left(Tr(P\Pi QG)\right).$$

Proof: This is equivalent to

$$\Im m\left(Tr(P\Pi PG)\right) = 0,$$

which follows from

$$Tr(P\Pi PG) = Tr(GP\Pi P) = \overline{Tr(P\Pi PG)}. \quad \square$$

Lemma 2.3 *For P, Q, Π, G as in lemma 2.2,*

$$Tr(P[\Pi,G]) = 2i\, \Im m\left(Tr(P\Pi QG)\right).$$

Proof:

$$\begin{aligned} Tr(P[\Pi,G]) &= Tr(P\Pi G) - Tr(PG\Pi) \\ &= Tr(P\Pi G) - \overline{Tr(\Pi GP)} \\ &= Tr(P\Pi G) - \overline{Tr(P\Pi G)} \\ &= 2i\, \Im m\, Tr(P\Pi G) \\ &= 2i\, \Im m\, Tr(P\Pi QG). \quad \square \end{aligned}$$

Lemma 2.4 *Let G and H be as in theorem 2.1. Let P be a spectral projection for H and $Q = I - P$, then*

$$Tr(P\,[G,[H,G]]) \;=\; 2\,Tr(PGQ[H,G]).$$

Proof:

$$\begin{aligned} Tr(P\,[G,[H,G]]) &= 2\,Tr(PGHG) - Tr(PG^2H) - Tr(PHG^2) \\ &= 2\,Tr(PGHG) - 2\,Tr(PG^2H) \\ &= 2\,Tr(PG[H,G]). \end{aligned}$$

Since $[G,[H,G]]$ is symmetric, the left side of this is real. This implies that $Tr(PG[H,G])$ is real but $i\,Tr(PG[H,G])$ is symmetric so by Lemma 2.2

$$\begin{aligned} i\,Tr(PG[H,G]) &= Im\,Tr(PG(i[H,G])) \\ &= Im\,Tr(PGQ(i[H,G])) \\ &= i\,Re\,Tr(PGQ[H,G]). \end{aligned}$$

But $Tr(PGQ[H,G])$ is real since it can be expressed as trace of the difference of two symmetric operators, $Tr(AHA^*) - Tr(A^*HA)$ where $A = PGQ$. □

Lemma 2.5 *Let H and G be as in Theorem 2.1 and let $P = P_{\leq n}$ be a spectral projection for H, then*

$$0 \leq Tr((\lambda_{n+1}I - H)PGQG) \;\leq\; Tr(PGQ[H,G]). \tag{2.5}$$

Proof: The right side of inequality (2.5) is

$$Tr(PGQ[H,G]) = Tr(PGQHG) - Tr(PGQGH) = Tr(PGQHG) - Tr(HPGQG),$$

and the first of these terms is bounded below as follows

$$\begin{aligned} Tr(PGQHG) &= Tr(PGQHQGP) \\ &\geq Tr(PGQ\lambda_{n+1}QGP) \\ &= Tr(\lambda_{n+1}PGQG). \end{aligned}$$

Finally, $Tr((\lambda_{n+1}I - H)PGQG) = Tr(QGP(\lambda_{n+1}I - H)PGQ) \geq 0$ since $(\lambda_{n+1}I - H) \geq 0$ on the range of $P_{\leq n}$. $\Box$

Proof of the theorem: for brevity we drop the subscript on $P_{\leq n}$.
Lemmas 2.4 and 2.5 imply

$$Tr\left((\lambda_{n+1}I - H)PG_jQG_j\right) \leq \frac{1}{2}\, Tr(P[G_j,[H,G_j]])$$

and by lemma 2.3

$$\sum_{j=1}^{m} Tr\left(P[\mathit{\Pi}_j, G_j]\right) = 2i\, \Im m \sum_{j=1}^{m} Tr\left(P\mathit{\Pi}_j QG_j\right).$$

Then,

$$\begin{aligned}
&\frac{\left|\sum_j Tr\left(P\left[\Pi_j, G_j\right]\right)\right|^2}{4} \\
&\leq \left|\sum_j Tr\left(P\Pi_j QG_j\right)\right|^2 \\
&= \left|\sum_j Tr\left(\left(\lambda_{n+1}I - H\right)^{1/2}\left(\lambda_{n+1}I - H\right)^{-1/2} P\Pi_j QG_j\right)\right|^2 \\
&= \left|\sum_j Tr\left(\left(\lambda_{n+1}I - H\right)^{-1/2} P\Pi_j QG_j\left(\lambda_{n+1}I - H\right)^{1/2} P\right)\right|^2 \\
&\leq \left|\sum_j Tr\left(\Pi_j P\left(\lambda_{n+1}I - H\right)^{-1} P\Pi_j\right)\right| \left|\sum_j Tr\left(QG_j\left(\lambda_{n+1}I - H\right)PG_jQ\right)\right| \\
&= \left|\sum_j Tr\left(\left(\lambda_{n+1}I - H\right)^{-1} P\Pi_j^2\right)\right| \left|\sum_j Tr\left(\left(\lambda_{n+1}I - H\right)PG_jQG_j\right)\right| \\
&\leq \frac{1}{2}\left(\sum_j Tr\left(\left(\lambda_{n+1}I - H\right)^{-1} P\Pi_j^2\right)\right)\left(\sum_j Tr\left(P\left[G_j,\left[H, G_j\right]\right]\right)\right).
\end{aligned}$$

Dividing both sides by $\frac{1}{2}\sum_j Tr(P[G_j,[H,G_j]])$ yields equation (2.1). $\Box$

As remarked above $(\lambda_{n+1} - H)$ is positive on the range of P so by the square root lemma (see [20]) the powers of this operator used above are all well-defined.

3 Applications

We begin by improving some of the bounds of [6].

Spherical domains

Let $\mathcal{M} = S^\nu \setminus B_\rho, \quad \nu \geq 2$, where B_ρ is a geodesic ball of radius $\rho > 0$. We assume the radius of S^ν is 1 (other radii are included by scaling). The Laplace–Beltrami operator on a spherical domain is the same as the angular momentum operator in quantum mechanics. It was shown in [6] that

$$C_{PPW} \leq \frac{16\left(1 + \dfrac{(\nu-2)\sin\rho}{2\,(1 - \cos\rho)\sqrt{\lambda_1}}\right)^2}{(1 - \cos\rho)^2\nu}.$$

Except in dimension 2, this has an unneeded and unpleasant factor, for we now have:

Corollary 3.1 *Let $\mathcal{M} = S^\nu \setminus B_\rho$, as above. Then,*

$$C_{HP} \leq \frac{16}{(1-\cos\rho)^2\nu}. \tag{3.1}$$

This constant diverges, as it must, when $\rho \to 0$, although not in the optimal way, and reduces to the Euclidean $C_{HP} = 4/\nu$ as $\mathcal{M}$ becomes small ($\rho \to \pi$).

Proof: The special choices are as follows. Embed S^ν in $\mathbf{R}^{\nu+1}$, and let G_j be the j^{th} stereographic coordinate, except for G_0, which is a dummy operator.

$$\begin{aligned} G_0 &:= 0, \\ G_j &:= \frac{x^j}{1-x^0} \quad \text{for } 1 \leq j \leq \nu \end{aligned}$$

where the Euclidean coordinates are denoted $(x^0, \ldots, x^\nu)$, with x^0 oriented towards the center of B_ρ.

Define

$$\Pi_j := -i\,\mathcal{R}\,\partial_j\,\mathcal{E},$$

where $\mathcal{E}$ is the extension of a function on S^ν to $\mathbf{R}^{\nu+1} \setminus \{0\}$ by writing a function f on S^ν in Euclidean coordinates with the restriction $\sum_{j=0}^{\nu}(x^j)^2 = 1$, and then letting

$$\mathcal{E}f(x^0,\dots,x^\nu) := f\left(\frac{x^0}{r},\dots,\frac{x^\nu}{r}\right), \quad \text{where } r := \left(\sum_{j=0}^{\nu}\left(x^j\right)^2\right)^{\frac{1}{2}}$$

and $\mathcal{R}$ is the restriction of a function in $\mathbf{R}^{\nu+1}$ to S^ν. Because of the embedding, we can calculate with the usual Euclidean Laplacian acting on functions on $\mathbf{R}^{\nu+1}$ independent of r. It is clear that $\sum_{j=0}^{\nu}\Pi_j^2 = -\Delta$.
The choice of the G's is equivalent to that in [6], where it was found that

$$\sum_{j=1}^{\nu}[G_j,[H,G_j]] = \sum_{j=1}^{\nu}2|\nabla G_j|^2 = \frac{2\nu}{(1-x^0)^2}\,.$$

The numerator in (2.2) is therefore bounded above by

$$\frac{4\nu n^2}{(1-\cos\rho)^2}\,.$$

Meanwhile, since Π_j satisfies Leibniz's rule for derivatives, we find that

$$\sum_{j=1}^{\nu}[\Pi_j,G_j] = -i\frac{\nu-1-x^0}{1-x^0} = -i\left(\frac{\nu-2}{1-x^0}+1\right)\,,$$

so

$$|\sum_j Tr(P_{\le n}[\Pi_j,G_j])|^2 \ge \left(\frac{\nu-2}{2}+1\right)^2 (TrP_{\le n})^2 = \left(\frac{\nu n}{2}\right)^2$$

yielding the inequality (3.1). $\square$

Hyperbolic domains

In [6] one of us derived an upper bound on C_{PPW} for subdomains of the two–dimensional hyperbolic space $\mathcal{H}^2$, but the bound diverges as the size of the domain becomes infinite. Unlike the case of spherical domains, this should not happen, since

$$-\Delta \ge \frac{1}{4} \tag{3.2}$$

in the sense of quadratic forms, which prevents λ_1 from approaching 0.

This drawback can be evaded by choosing the auxiliary operators from a semigeodesic (Fermi) coordinate system. Recall that there is a semigeodesic coordinate system (t, r) for $\mathcal{H}^2$ with the metric $ds^2 = dt^2 + cosh^2 t\, dr^2$ (cf. [5], p. 263, where, however, there are some misprints).

We shall choose $G_1 := t$, $G_2 := r$, and $\Pi_j := i[H, G_j] = i(-\Delta G_j - 2\nabla G_j \cdot \nabla)$ for $j = 1, 2$. Calculations with equation (1.1) readily show that

$$
\begin{aligned}
\Delta t &= \tanh(t)\,, \\
\Delta r &= 0\,, \\
2\nabla t \cdot \nabla \zeta &= 2\,\frac{\partial \zeta}{\partial t}\,, \\
2\nabla r \cdot \nabla \zeta &= \frac{2}{\cosh^2(t)}\,\frac{\partial \zeta}{\partial r}\,.
\end{aligned}
\tag{3.3}
$$

Corollary 3.2 *Suppose that* $\mathbf{\Omega}$ *is a domain in* $\mathcal{H}^2$ *such that the distance from any point of* $\mathbf{\Omega}$ *to* $\{t = 0\}$ *is at most* T*. Then*

$$C_{HP} \leq \frac{4\,e^{2T}}{1 + \cosh^2(T)}.$$

Remark: Because of the freedom to choose the orientation of the Fermi coordinate system, T is intuitively an upper bound on the thinner dimension of $\mathbf{\Omega}$.

Proof: Because of (3.3),

$$\|\Pi_1 u\|^2 + \|\Pi_2 u\|^2 = \int_\Omega \left(\left(\tanh(t) u + 2\frac{\partial u}{\partial t} \right)^2 + \frac{4}{\cosh^4(t)} \left(\frac{\partial u}{\partial r} \right)^2 \right) dV.$$

Since

$$\|\nabla u\|^2 = \int_\Omega \left(\left(\frac{\partial u}{\partial t} \right)^2 + \frac{1}{\cosh^2(t)} \left(\frac{\partial u}{\partial r} \right)^2 \right) dV,$$

we get

$$\|\Pi_1 u\|^2 + \|\Pi_2 u\|^2 \leq \int_\Omega \left(\tanh^2(T) u^2 + 4 \tanh(T) u\, |\nabla u| + 4\, |\nabla u|^2 \right) dV.$$

Because of (3.2) and the Cauchy-Schwarz inequality, this is bounded above by

$$\tanh^2 T\|u\|^2 + 4\tanh T\int_\Omega |u|\,|\nabla u|\,dV + 4\|\nabla u\|^2 \le 4\left(1+\tanh T\right)^2\|\nabla u\|^2.$$

In other words, we can take $\beta = 4\left(1+\tanh T\right)^2$.

According to (2.4) and (3.3),

$$C_{HP} \le \frac{8n\ \left(1+\tanh T\right)^2}{2\,Tr\Big(P_{\le n}\left(1+\frac{1}{\cosh^2 t}\right)\Big)} \le \frac{4\left(\cosh T + \sinh T\right)^2}{1+\cosh^2 T}.$$

This simplifies to the statement of the corollary. □

As $T \to 0$, this reduces to the Euclidean bound 2 but as $T \to \infty$ then $C_{HP} \le 16$.

More general manifolds

In [6] one of us produced upper bounds on C_{PPW} for manifolds admitting a global semigeodesic (Fermi) coordinate system. These bounds reflected the curvature of $\mathcal{M}$ (specifically, the Gauss curvature in two dimensions and the sectional curvature in higher dimensions). Here we remark briefly on the extensions of those bounds using the main theorem of this article, with auxiliary operators satisfying (2.3). As remarked earlier, this leads to a simplification, which we formalize as follows:

Corollary 3.3 *Let G_j be real C^2 functions, and define the corresponding Π_j by* (2.3). *Suppose that γ and β are constants such that*

$$-\sum_{j=1}^m [H, G_j]^2 \le \beta H\,,$$

then

$$C_{HP} \le \frac{2n\beta}{\left|\sum_{j=1}^m Tr\left(P_{\le n}\left[G_j, [H, G_j]\right]\right)\right|}.$$

In particular, the bounds on C_{PPW} from section 4 of [6] all apply to C_{HP}. As a representative, we cite a two-dimensional corollary slightly extending a result of that paper:

Corollary 3.4 *Let the dimension $\nu = 2$ and suppose that $\mathbf{\Omega}$ has a semigeodesic coordinate system with geodesic coordinate x^1, chosen so that $\mathcal{P} := \{x^1 = 0\}$*

intersects $\mathbf{\Omega}$. *Let* $D := \sup\left(\operatorname{dist}(\mathcal{P},\partial\mathbf{\Omega})\right)$ *and suppose that the curvature* h_1 *of* $\mathcal{P}$ *and the Gauss curvature* κ *of* $\mathbf{\Omega}$ *are bounded above and below by constants:*

$$\begin{aligned} h_- &\leq h_1(0, x_2, \ldots) \leq h_+\,, \\ \kappa_- &\leq \kappa(x_1, x_2, \ldots) \leq \kappa_+\,. \end{aligned}$$

Then

$$C_{HP} \leq \left(2 + \frac{\sup\left(|r(x^1, a, k)| : |x_1| \leq D, a = h_\pm, k = \kappa_\pm\right)}{\sqrt{\lambda_1}}\right)^2 .$$

where r *is the function written explicitly in* [6] *as*

$$r(s,a,k) = \begin{cases} \dfrac{a - \sqrt{k}\tan\left(\sqrt{k}\, s\right)}{1 + \frac{a}{\sqrt{k}}\tan\left(\sqrt{k}\, s\right)}, & k > 0 \\[2ex] \dfrac{a + \sqrt{|k|}\tanh\left(\sqrt{|k|}\, s\right)}{1 + \frac{a}{\sqrt{|k|}}\tanh\left(\sqrt{|k|}\, s\right)}, & k < 0 \\[2ex] \dfrac{a}{1 + as}, & k = 0\,. \end{cases}$$

Homogeneous spaces and minimally immersed submanifolds

In this section we study eigenvalue differences for some special manifolds without boundaries. For background material see [13], [5], and [3]. A complicating feature here is that the lowest eigenvalue $\lambda_0 = 0$ automatically. We number the eigenvalues so that $0 < \lambda_1$ is the first non-trivial eigenvalue. Because of this, the appropriate gap bounds will be of the form

$$\lambda_{n+1} - \lambda_n \leq \lambda_1 + C\langle\lambda_\ell\rangle_{\ell\leq n}\,.$$

And we define a modified universal constant

$$C'_{PPW} := \sup_{n,\mathbf{\Omega}\subset\mathcal{M}} \frac{\lambda_{n+1} - \lambda_n - \lambda_1}{\langle\lambda_\ell\rangle_{\ell\leq n}}\,.$$

The analogous Hile–Protter quantity will also gain a term with λ_1 :

$$C'_{HP} := \sup_{n,\Omega\subset\mathcal{M}} \frac{1-\lambda_1\langle\frac{1}{\lambda_{n+1}-\lambda_\ell}\rangle_{\ell\le n}}{\langle\frac{\lambda_\ell}{\lambda_{n+1}-\lambda_\ell}\rangle_{\ell\le n}} .$$

As before $C'_{PPW} \le C'_{HP}$.

We are interested in studying the eigenvalues of the problem

$$-\Delta\phi = \lambda\phi \quad \text{on} \quad \mathcal{M}$$

where $\mathcal{M}$ is a Riemannian manifold, without boundary, of finite volume V. Two of the more important results in this direction are the bounds of Yang and Yau [21] for minimally immersed submanifolds of $S^N(1)$, and Li [14] for compact homogeneous spaces. For details about these spaces, see [5], [16], and [9]. In our algebraic approach, the key point is that symmetry forces certain special properties on the eigenfunctions and eigenvalues, which we exploit in our choices of the operators G and Π.

In [21], Yang and Yau use the fact that the coordinate functions on a compact minimally immersed submanifold $\mathcal{M}^m \subset S^N(1)$ are eigenfunctions of $-\Delta$ with degenerate eigenvalue m = dimension of $\mathcal{M}^m$, to obtain a bound on the gap between successive higher eigenvalues. In particular they show, as corrected by Leung in [15]

$$\lambda_{n+1}-\lambda_n \le m + \frac{2}{m(n+1)}\left(\sqrt{\Lambda^2+m^2\Lambda(n+1)}+\Lambda\right) \tag{3.4}$$

where $\Lambda = \sum_{\ell=0}^{n}\lambda_\ell$.

Leung takes up the same problem in [15] and uses the method of Hile and Protter (see [10]) to obtain an improved but very complicated looking version of (3.4).

In [14], Li offers a theorem for more general homogeneous manifolds:

Lemma 3.5 (Li, Proposition 1 and proof of Proposition 1 of [14]) *Let $\mathcal{M}$ be a compact homogeneous manifold and take $\{\phi_{1,\alpha}\}_{\alpha=1}^{k}$ to be an orthonormal basis for the k-dimensional eigenspace of λ_1 (the first non-zero eigenvalue of $-\Delta$). Then,*

$$\sum_{\alpha=1}^{k}\phi_{1,\alpha}^2 = \frac{k}{V} \quad \text{and} \quad \sum_{\alpha=1}^{k}|\nabla\phi_{1,\alpha}|^2 \le \frac{\lambda_1 k}{V}. \tag{3.5}$$

Using these results, Li is able to prove that the eigenvalues of $-\Delta$ satisfy

$$\lambda_{n+1} - \lambda_n \leq \frac{2}{n+1}\left(\sqrt{\Lambda^2 + (n+1)\Lambda\lambda_1} + \Lambda\right) + \lambda_1 \tag{3.6}$$

where $\lambda_{n+1} > \lambda_n > \lambda_k$. It is interesting to note that this bound does not depend explicitly on the dimension of the space.

We now obtain a corollary to Theorem 2.1 which we will use to improve the bounds of Yang and Yau (3.4) and Li (3.6).

Corollary 3.6 *Suppose $\{\phi_{1,\alpha}\}_{\alpha=1}^k$ are k functions from the eigenspace of $\lambda_1 > 0$ of $-\Delta$ on a manifold $\mathcal{M}$. If, in addition, there exist constants $a, b > 0$ such that*

$$\sum_{\alpha=1}^{k} \phi_{1,\alpha}^2 = a \quad \text{and} \quad \sum_{\alpha=1}^{k} |\nabla\phi_{1,\alpha}\cdot\nabla\psi|^2 \leq b\,|\nabla\psi|^2,$$

then the eigenvalues of $-\Delta$ satisfy

$$\left\langle \frac{\lambda_1^2 a + 4b\lambda_\ell}{\lambda_{n+1} - \lambda_\ell} \right\rangle_{\ell\leq n} \geq \lambda_1 a\,.$$

Proof: The special choices to be made in Theorem 2.1 are

$$G_\alpha = \phi_{1,\alpha} \quad \text{for} \quad \alpha = 1, \ldots, k$$

and

$$\Pi_\alpha = i[H, G_\alpha] = i(-\Delta G_\alpha - 2\nabla G_\alpha\cdot\nabla).$$

First observe that if G is a function in $L^2(\mathcal{M})$ and $H = -\Delta$,

$$[G,[H,G]] = 2\nabla G\cdot\nabla G = \Delta(G^2) - 2G(\Delta G). \tag{3.7}$$

With the above choices for G_α and Π_α the right side of inequality (2.1) is

$$\begin{aligned}
\frac{\left|\sum_{\alpha=1}^k Tr\left(P_{\leq n}[[H,G_\alpha],G_\alpha]\right)\right|^2}{2\sum_{\alpha=1}^k Tr\left(P_{\leq n}[G_\alpha,[H,G_\alpha]]\right)} &= \frac{1}{2}\sum_{\alpha=1}^{k} Tr\left(P_{\leq n}\left(\Delta\left(G_\alpha^2\right) - 2G_\alpha\left(\Delta G_\alpha\right)\right)\right) \\
&= \frac{1}{2}Tr\left(P_{\leq n}\left(\Delta\left(\sum_{\alpha=1}^{k} G_\alpha^2\right) + 2\sum_{\alpha=1}^{k} G_\alpha^2\lambda_1\right)\right) \\
&= Tr\left(a\lambda_1 P_{\leq n}\right) = a\lambda_1(n+1).
\end{aligned}$$

We now obtain an upper bound for the left side of (2.1). We have

$$\begin{aligned}
&\sum_{\ell=0}^{n}\sum_{\alpha=1}^{k}(\lambda_{n+1}-\lambda_\ell)^{-1}\,\|[H,G_\alpha]\phi_\ell\|^2 \\
&\qquad = \sum_{\ell=0}^{n}(\lambda_{n+1}-\lambda_\ell)^{-1}\int\sum_{\alpha=1}^{k}(\lambda_1 G_\alpha\phi_\ell-2\nabla G_\alpha\cdot\nabla\phi_\ell)^2 \\
&\qquad = \sum_{\ell=0}^{n}(\lambda_{n+1}-\lambda_\ell)^{-1}\left(\lambda_1^2 a+4\int\sum_{\alpha=1}^{k}|\nabla G_\alpha\cdot\nabla\phi_\ell|^2\right) \\
&\qquad \leq \sum_{\ell=0}^{n}(\lambda_{n+1}-\lambda_\ell)^{-1}\left(\lambda_1^2 a+4b\lambda_\ell\right).
\end{aligned}$$

This proves the corollary. □

This leads us to

Corollary 3.7 *Let $\mathcal{M}^m$ be a minimally immersed submanifold of $S^N(1)$, then*

$$C'_{PPW}\leq C'_{HP}\leq\frac{4}{m}\,.$$

Proof: Choose $\{\phi_{1,\alpha}\}_{\alpha=1}^{k}$ to be the coordinate functions of $\mathbf{R}^{N+1}$. Then $\lambda_1=m$, $k=N+1$, $a=1$, and $b=1$ (see [5], [15] and [21]) in Corollary 3.6. This yields

$$\sum_{\ell=0}^{n}\frac{m^2+4\lambda_\ell}{\lambda_{n+1}-\lambda_\ell}\geq m(n+1)\,.$$

From which the corollary follows. □

Corollary 3.8 *If $\mathcal{M}$ is a compact homogeneous manifold, then*

$$C'_{PPW}\leq C'_{HP}\leq 4\,.$$

Proof: Choose $\{\phi_{1,\alpha}\}_{\alpha=1}^{k}$ as in Lemma 3.5. Then, by the result of the lemma, $a=\frac{k}{V}$ and $b=\lambda_1 a$. This implies

$$\sum_{\ell=0}^{n}\frac{\lambda_1+4\lambda_\ell}{\lambda_{n+1}-\lambda_\ell}\geq n+1\,,$$

which yields the result. □

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